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DYNKIN DIAGRAMS AND INTEGRABLE MODELS BASED ON LIE SUPERALGEBRAS

JONATHAN M. EVANS¹

*DAMTP, University of Cambridge
 Silver Street, Cambridge CB3 9EW, UK
 J.M.Evans@damtp.cam.ac.uk*

JENS OLE MADSEN

*Laboratoire de Physique Théorique
 ENSLAPP², Groupe d'Annecy
 Chemin de Bellevue, F-74941 Annecy le Vieux, France
 madsen@lapphp.in2p3.fr*

ABSTRACT

An analysis is given of the structure of a general two-dimensional Toda field theory involving bosons and fermions which is defined in terms of a set of simple roots for a Lie superalgebra. It is shown that a simple root system for a superalgebra has two natural bosonic root systems associated with it which can be found very simply using Dynkin diagrams; the construction is closely related to the question of how to recover the signs of the entries of a Cartan matrix for a superalgebra from its Dynkin diagram. The significance for Toda theories is that the bosonic root systems correspond to the purely bosonic sector of the integrable model, knowledge of which can determine the bosonic part of the extended conformal symmetry in the theory, or its classical mass spectrum, as appropriate. These results are applied to some special kinds of models and their implications are investigated for features such as supersymmetry, positive kinetic energy and generalized reality conditions for the Toda fields. As a result, some new families of integrable theories with positive kinetic energy are constructed, some containing a mixture of massless and massive degrees of freedom, others being purely massive and supersymmetric, involving a number of coupled sine/sinh-Gordon theories.

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1. Introduction

Toda theories provide a uniform way of constructing both massless and massive integrable field theories from Lie algebras [1–4]. In recent years these models have received considerable attention. It is well known that those based on finite-dimensional Lie algebras exhibit extended conformal symmetry (see [5–7] for reviews) whilst those based on affine Kac-Moody algebras are massive integrable field theories for which exact S-matrices can be found (see eg. [8]). Toda theories based on Lie superalgebras have also been considered by a number of authors [9–26]. These integrable models contain both bosonic and fermionic fields, but they need not be supersymmetric in general. They have been studied principally because they provide field-theoretic realizations of \mathcal{W} -algebras, [13–18] but there has also been work concerned with S-matrices for massive models [20–26]. Results for theories based on superalgebras are much less complete than in the bosonic case, however.

In this paper we establish a number of results concerning the structure of a Toda theory based on a general Lie superalgebra. We show that the bosonic sector of any such theory consists of a sum of bosonic Toda models whose lagrangians can contribute with positive or negative signs and we describe a simple way of reading off the Dynkin diagrams corresponding to these bosonic sub-theories from the Dynkin diagram of the original Lie superalgebra. By applying this idea to a conformally-invariant theory one can easily determine the bosonic subalgebra of the extended conformal symmetry which is responsible for the integrability of the model [5–7], while for a massive theory one can find the classical mass spectrum in terms of the known mass spectra of bosonic models [3,8,27]. In addition to this information on the bosonic content of the theory, we give a criterion for the decoupling of certain fermionic degrees of freedom. After explaining these general results, we discuss various classes of examples.

Almost all Toda models based on finite-dimensional Lie superalgebras are conformally invariant, in close analogy with the bosonic situation (the exceptions are discussed in sections 4 and 6). For models based on an infinite-dimensional superalgebra, however, one finds behaviour which differs markedly from the bosonic case. While bosonic affine Toda models are massive integrable field theories, we shall show that the bosonic sector of a theory based on an affine superalgebra can in general be a combination of massive and conformal bosonic Toda theories, typically including a number of free massless bosons as well. Our methods will allow us to identify those models which are purely massive.

Another way in which superalgebra theories differ from their bosonic counterparts is that their kinetic energy can have indefinite signature. While this is not necessarily a fatal flaw, particularly in the context of conformal field theory where the study of non-unitary models is common, it is clearly interesting to ask which models have conventional, positive-definite kinetic energy. This point was originally raised by Olshanetsky [9], but in fact the class of Toda models he considered can be generalized to allow for non-standard or *twisted* reality conditions on the fields [14,19]. Our techniques enable one to see easily how such reality conditions descend to the bosonic sector of a superalgebra theory. By extending the analysis in this way, we find new families of integrable field theories with positive-definite lagrangians. Most of these contain both massless and massive degrees of freedom, but several examples are purely massive and supersymmetric, consisting of sine/sinh-Gordon theories coupled together in various ways.

The Dynkin diagram construction at the heart of this paper is intimately related to the question of how one recovers a Cartan matrix from a Dynkin diagram for a Lie superalgebra, including the rather subtle issue of reconstructing the correct relative signs for all the entries. We discuss this in detail in the next section. We then move on to Toda theories proper and to the topics indicated above. Throughout the paper we consider only *classical, abelian* Toda theories with *real* coupling constants, but we shall draw attention at various stages to those theories that would be particularly interesting to investigate further at the quantum level.

2. Simple roots and Dynkin diagrams

We shall be concerned with Lie superalgebras $\mathcal{G}(\mathcal{R})$ defined in terms of some finite set of simple roots \mathcal{R} . This means we consider precisely the superalgebras of types A, B, C, D, E, F, G and their affine extensions, as defined by the Cartan-Kac classification. We first recall some elementary facts concerning these algebras; for more details, see [28–37,10].

The simple roots may be identified with a set of n vectors α_i living in an r -dimensional space V , with $i \in \{1, \dots, n\}$ (for affine algebras we project onto a horizontal subspace). Each simple root is graded bosonic (even) or fermionic (odd) as specified by the disjoint union $\mathcal{R} = \mathcal{B} \cup \mathcal{F}$ or equivalently by values of the labels $i \in \varepsilon$ and $i \in \tau$ respectively, where ε and τ are complementary subsets of $\{1, \dots, n\}$. We denote the span of any set of vectors \mathcal{S} in V by $\langle \mathcal{S} \rangle$. The simple roots span the space, so $V = \langle \mathcal{R} \rangle$ and $n \geq r$ (in fact $n = r$ for finite-dimensional superalgebras and $n = r+1$ for infinite-dimensional superalgebras, with the exceptions of $A(m, m)$ and $A(m, m)^{(1)}$ which have $n = r+1$ and $n = r+2$ respectively.) Every superalgebra $\mathcal{G}(\mathcal{R})$ contains a unique maximal bosonic subalgebra which we denote by $\mathcal{G}(\mathcal{R})_{\text{bos}}$, but this depends on the entire root system \mathcal{R} and not just on the bosonic simple roots \mathcal{B} .

There is a natural inner-product on V which will be denoted by a dot and which is inherited from the invariant inner-product on the algebra $\mathcal{G}(\mathcal{R})$. The inner-product is non-degenerate on V , by construction (since we take projected roots for affine algebras). For bosonic algebras the inner-product is positive-definite but for Lie superalgebras it is typically of indefinite signature, which gives rise to some very important differences. If U is any subspace of V and U^\perp denotes the orthogonal subspace, then $U \cap U^\perp \neq \{0\}$ in general. If the restriction of the inner-product to U is non-degenerate, however, then $U \cap U^\perp = \{0\}$ and $V = U \oplus U^\perp$.

The precise definition of $\mathcal{G}(\mathcal{R})$ can be given in terms of sets of generators H_i and E_i^\pm corresponding to the simple roots. These must satisfy the basic relations

$$[H_i, E_j^\pm] = \pm k_{ij} E_j^\pm, \quad [E_i^+, E_j^-] = \delta_{ij} H_i. \quad (2.1)$$

The Cartan matrix k_{ij} has dimension $n \times n$, rank r , and it encodes all necessary information about the root system. In these equations, no summation is implied and the bracket is understood to be graded. The definition of the algebra must be completed by further conditions on the generators known as Serre relations [28,30,31,32,37].

It is clear that by re-scaling the generators H_i and E_i^\pm we can change the Cartan matrix $k_{ij} \rightarrow \lambda_i k_{ij}$ (no summation) where λ_i are any non-zero constants. Using this freedom, it is always possible to make the Cartan matrix symmetric; it is then just the

matrix of inner-products of the simple roots, which we write as $k_{ij} = a_{ij} = \alpha_i \cdot \alpha_j$. The matrix a_{ij} is unique up to a permutation of the simple roots and an overall constant. We can take its entries to be integers, except for $D(2, 1; \alpha)$ and its affine extensions at generic values of the parameter α . To define Dynkin diagrams, it is convenient to introduce a slightly different Cartan matrix which we write as $k_{ij} = b_{ij} = \lambda_i a_{ij}$, where $\lambda_i > 0$. When the entries a_{ij} are integers, we can choose λ_i so that b_{ij} are also integers which obey: **(i)** $|b_{ii}| = 2$ or 0 ; **(ii)** if $b_{ii} = 0$ then b_{ij} ($j = 1, \dots, n$) have no common factor. These conditions fix the normalization of each row, so that b_{ij} is unique up to an overall sign. It is also useful to note the properties: **(iii)** b_{ij} and b_{ji} have the same sign; **(iv)** if $b_{ij} \neq b_{ji}$ then either $|b_{ij}| = 1$ or $|b_{ji}| = 1$. We can define b_{ij} satisfying (i), (iii) and (iv) for the exceptional D algebras too, but we must replace (ii) with some other criterion for normalizing the rows corresponding to null roots if b_{ij} is to be unique. We emphasize that with our conventions the diagonal entries of a_{ij} and b_{ij} can be positive or negative. There are a number of other choices for the Cartan matrix adopted in [10,30-37], many of which rescale the rows so that the diagonal entries are non-negative, so relinquishing property (iii).

For any bosonic Lie algebra, the simple root system \mathcal{R} is essentially unique: any two simple root systems are related by the action of the Weyl group and the Cartan matrices coincide up to a permutation of rows and columns [28,30]. For a Lie superalgebra, however, it is quite possible to find inequivalent simple root systems \mathcal{R} and \mathcal{R}' which lead to isomorphic algebras $\mathcal{G}(\mathcal{R}) = \mathcal{G}(\mathcal{R}')$ despite the fact that their Cartan matrices are completely different. It is possible to reconstruct all allowed simple root systems for a given superalgebra from any one simple root system by applications of transformations which generalize the reflections contained in the bosonic Weyl group; for more details see [10,36,37]. We should merely be aware that for a given algebra there may be a number of equally valid simple roots systems \mathcal{R} .

We now give rules for encoding the Cartan matrix b_{ij} in a Dynkin diagram—these follow very closely the usual bosonic conventions.

- Draw a *node* for each simple root α_i , coloured white if it is bosonic ($i \in \varepsilon$), black if it is fermionic and non-null ($i \in \tau$ and $\alpha_i^2 \neq 0$), grey or hatched if it is fermionic and null ($i \in \tau$ and $\alpha_i^2 = 0$).
- Draw a *bond* between the nodes corresponding to i and j consisting of $\max(|b_{ij}|, |b_{ji}|)$ lines, this being the *strength* of the bond, with an arrow from i to j if $|b_{ji}| > |b_{ij}|$.

We have assumed here that the entries of the Cartan matrix are integers. We can extend the rules to encompass those members of the continuous family $D(2, 1; \alpha)$ and their affine counterparts whose Cartan matrices have non-integer entries. We proceed as before, but instead of drawing a bond consisting of a number of lines to represent an integer, we draw a single line and label it with the real number b_{ij} whenever necessary; see eg. [36,37].

The scheme above defines unambiguously how one draws a Dynkin diagram to represent a given Cartan matrix. Although our choice for the Cartan matrix itself may differ in small ways from some others in the literature, the Dynkin diagrams which follow from the rules we have given agree with those which can be found in standard references such as [31,32,36,37]. So far everything seems like a simple modification of the bosonic situation. The big difference, however, is that one cannot uniquely reconstruct the Cartan matrix for a superalgebra from its Dynkin diagram without giving some additional rules. The problematical thing is how to recover the *signs* of the entries of the Cartan matrix. We have chosen to focus on a_{ij} and b_{ij} because they are the Cartan matrices which are most immediately relevant for constructing integrable lagrangians, but also because the signs of

their entries can be recovered in a relatively straightforward way. To explain how, it is best to describe in more detail what kinds of Cartan matrices and diagrams are actually allowed by the classification of [31-35].

It will be convenient to refer to the infinite series of superalgebras of types A, B, C, D as *classical* (so this term is being used here in a different sense than in [31,37]) and to distinguish them from the *exceptional* algebras G_2 , F_4 , E_6 , E_7 , E_8 , $D(2,1;\alpha)$, $G(3)$, $F(4)$, all their affine extensions, and $D_4^{(3)}$. The Dynkin diagrams for the finite-dimensional, classical bosonic Lie algebras of types A, B, C and D are shown below in Figure 1.

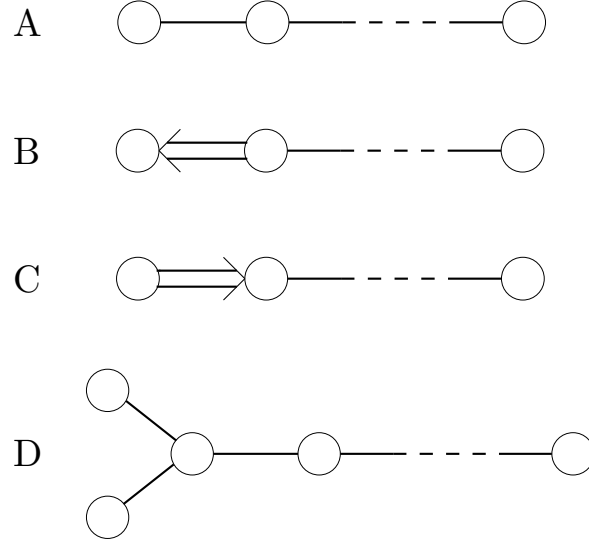


Figure 1: classical bosonic Dynkin diagrams

In a similar spirit, let us introduce the symbols A^* , B^* and D^* corresponding to the fermionic diagrams in Figure 2.

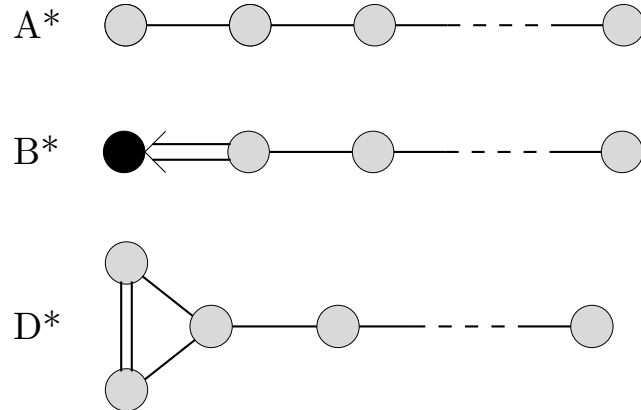


Figure 2: classical fermionic Dynkin diagrams

All simple root systems for classical Lie superalgebras can be obtained by combining the bosonic diagrams in Figure 1 with the fermionic diagrams in Figure 2. Any Dynkin diagram for a finite-dimensional, classical superalgebra consists of a chain of alternating blocks of type A and A^* which may in addition have a tail of type B, B^* , D, D^* , or C at one end

only. Any Dynkin diagram for an infinite-dimensional, classical superalgebra consists of a chain of alternating blocks of type A and A* which either has its ends joined to form a circle, this being permissible if the total number of grey nodes is even, or else has tails of type B, B*, D, D* or C attached at both ends. Any such diagrams are allowed.

We will make much further use of the idea that the diagrams in Figures 1 and 2 are building blocks for general Dynkin diagrams. In particular, this suggests an obvious notation for various other classes of diagrams. For instance, the Dynkin diagrams for the classical, bosonic affine algebras can be denoted:

Algebra	Type of Diagram
$A_{m-1}^{(1)}$	\hat{A} m nodes
$B_m^{(1)}$	B-D $m+1$ nodes
$C_m^{(1)}$	C-C $m+1$ nodes
$D_{m+1}^{(1)}$	D-D $m+2$ nodes
$A_{2m-2}^{(2)}$	B-C m nodes
$A_{2m-1}^{(2)}$	C-D $m+1$ nodes
$D_{m+1}^{(2)}$	B-B $m+1$ nodes

where $m \geq 2$ (compare with Figures 3 and 5 of [29] or Tables Aff1 and Aff2 of [30]). The symbol \hat{A} indicates a diagram of type A whose ends have been joined to form a circle, while the other symbols indicate a joining of two diagrams of type B, C or D. (Note that when there are just two nodes \hat{A} represents the same diagram as B-B or C-C; when there are three nodes B-D coincides with C-C, and C-D coincides with B-B; when there are four nodes \hat{A} coincides with D-D.) We shall use a similar notation when we come to discuss fermionic diagrams, and in preparation we introduce two further classes B* and D* which contain both bosonic and fermionic nodes, as shown in Figure 3. Although these are composites of diagrams already defined, they will prove useful later.

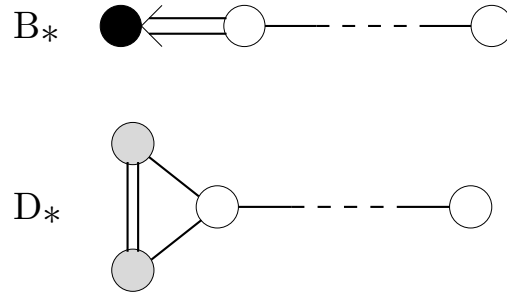


Figure 3: two additional families of classical Dynkin diagrams

We now supply the final ingredient in the correspondence between simple root systems and Dynkin diagrams for classical algebras by describing how to reconstruct the signs of the entries in a Cartan matrix from a diagram. For bosonic or fermionic simple roots with non-vanishing length-squared this is straightforward: in any connected part of a diagram which involves only white or black nodes, these nodes all represent numbers with the same sign, and the bonds attached to them all represent numbers with the opposite sign. For diagrams with null fermionic roots things are more complicated, however, and the relative signs must be taken in accordance with those written in Figure 4. (The associated bosonic

systems \mathcal{B}_\pm will be introduced shortly.) The signs on the dotted bonds of the fermionic diagrams indicate the choices necessary for an additional bond which appears in forming any larger, composite diagram. Notice that these rules could not be applied consistently to a circle containing an odd number of grey nodes; but, as mentioned above, such diagrams are forbidden by the classification of superalgebras.

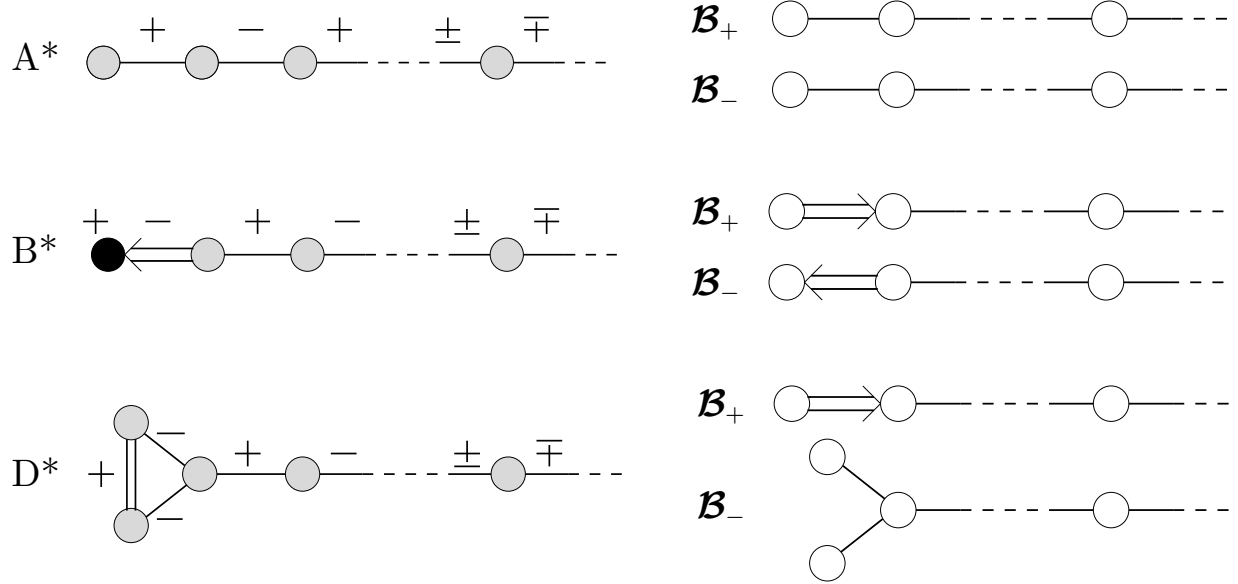


Figure 4: signs associated with fermionic Dynkin diagrams

It would be nice to give a set of unambiguous rules for reconstructing such signs from any Dynkin diagram. Unfortunately, this is not possible if one adheres to the standard way of drawing Dynkin diagrams—which is now more or less universally followed—because there is an inherent ambiguity. The difficulty arises with diagrams of the type shown in Figure 5, where the dotted bonds can be of various of strengths.

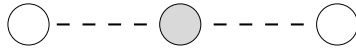


Figure 5: ambiguous Dynkin diagrams

If these bonds are of strength one, for instance, the diagram could represent root systems for either of the algebras $D(2,1)$ or $A(1,1)$, depending on whether we take the signs corresponding to the bonds to be equal or opposite. This means that there is not quite a one-to-one correspondence between diagrams and simple roots systems for superalgebras (and this is quite distinct from the fact that many different simple root systems can lead to isomorphic algebras). In practice, however, this ambiguity is not very serious because it involves essentially only this one particular class of diagrams.

It is because of this complication that we have chosen to present things initially in a rather concrete way. We have also concentrated on the classical algebras because we shall not have much reason to deal directly with the exceptional algebras in this paper. Nevertheless, now that we have explained the difficulties, we can give a set of rules which is as clear-cut as possible and which allows one to recover the signs corresponding to any

Cartan matrix, including those for exceptional algebras. These rules apply to *any Dynkin diagram* or *subdiagram* (obtained by deleting a number of nodes and their attached bonds).

- For any white or black node in any diagram, the sign on the node is opposite to the sign on any bond attached to it.
- For any diagram without branches (ie. consisting of a line or a circle of nodes) the signs on two bonds attached to any given grey node must be opposite to one another, with the following exceptions:
 - (i) diagrams of the form shown in Figure 5, for which the signs may be equal or opposite;
 - (ii) triangles of three grey nodes, which are dealt with by the next rule.
 (These exceptions involve members of the family $D(2,1;\alpha)$ for various values of α .)
- For any diagram consisting of a triangle of three grey nodes, the signs on the bonds must be such that the sum of the strengths of the bonds, taken with these signs, is zero.

It is easy to see that these rules hold for all the classical examples given in Figure 4, and indeed that the signs given there are necessary consequences of these rules. The exceptional algebras $G(3)$, $F(4)$ and their affine extensions can be considered similarly; the Dynkin diagrams can be found in [36].

Now that we have clarified some aspects of the relationship between Cartan matrices and Dynkin diagrams, we consider a general simple root system $\mathcal{R} = \mathcal{B} \cup \mathcal{F}$ and introduce two associated bosonic simple root systems \mathcal{B}_\pm .

Definition: $\mathcal{B}_\pm = \{ \alpha_i \in \mathcal{B} \mid \alpha_i^2 \gtrless 0 \} \cup \{ \alpha_i + \alpha_j \mid \alpha_i, \alpha_j \in \mathcal{F} \mid (\alpha_i + \alpha_j)^2 \gtrless 0 \}$.

The sets \mathcal{B}_\pm are clearly contained within the bosonic subsystem of the entire root system of $\mathcal{G}(\mathcal{R})$. If we consider the bosonic subalgebras $\mathcal{G}(\mathcal{B}_\pm)$ which are generated by \mathcal{B}_\pm , these must be contained within $\mathcal{G}(\mathcal{R})_{\text{bos}}$. The following observation will play a central role in all the following developments.

Proposition 1: Let \mathcal{R} be any superalgebra simple root system with \mathcal{B}_\pm defined as above.

- (a) The sets \mathcal{B}_\pm are mutually orthogonal and the restrictions of the inner-product on $\langle \mathcal{R} \rangle$ to the subspaces $\langle \mathcal{B}_\pm \rangle$ are positive/negative-definite. Consequently $\mathcal{G}(\mathcal{B}_\pm)$ commute and we have a direct sum of Lie algebras $\mathcal{G}(\mathcal{B}_+) \oplus \mathcal{G}(\mathcal{B}_-) \subset \mathcal{G}(\mathcal{R})_{\text{bos}}$.
- (b) If $i, j \in \tau$ then $\alpha_i + \alpha_j \in \mathcal{B}_\pm$ iff $\alpha_i \cdot \alpha_j \gtrless 0$.

Proof: (a) The fact that the algebras commute follows immediately from the property that the sets of roots \mathcal{B}_\pm are mutually orthogonal, which can be established by inspection. On a simple bosonic Lie algebra, the invariant inner-product is essentially unique and must have definite signature. Hence, we need only check that the inner-product is positive or negative on any one simple root in each simple factor. It follows from the definitions of \mathcal{B}_\pm that it must be positive/negative-definite on $\langle \mathcal{B}_\pm \rangle$. Part (b) is immediate if the roots are both null, but it is also easy to check that it holds for the combination of a grey and a black node, or two black nodes, which occur only in (sub)diagrams of type B^* .

The significance of the sets \mathcal{B}_\pm will become clearer in the next section. Notice that part (b) of Proposition 1 allows the sets of roots \mathcal{B}_\pm to be read off very easily from the Dynkin diagram for \mathcal{R} with the pattern of relative \pm signs attached according to Figure 4. Since this observation proves so useful, we have recorded the results for \mathcal{B}_\pm in Figure 4 for the primitive kinds of fermionic diagrams A^* , B^* and D^* . These results can easily be combined to find \mathcal{B}_\pm for more complicated root systems \mathcal{R} . In general the diagrams for \mathcal{B}_\pm need not be connected and, correspondingly, the algebras $\mathcal{G}(\mathcal{B}_\pm)$ may consist of direct sums of various simple factors.

3. Simple root systems and integrable models

First we establish some conventions. Two-dimensional super-Minkowski space has bosonic light-cone coordinates x^\pm and fermionic light-cone coordinates θ^\pm which are real (or Majorana) spinors. Lorentz transformations act on these coordinates according to $x^\pm \rightarrow e^{\pm\lambda}x^\pm$ and $\theta^\pm \rightarrow e^{\pm\lambda/2}\theta^\pm$ where λ is the rapidity. Covariant super-derivatives obey

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\theta^\pm\partial_\pm, \quad D_\pm^2 = -i\partial_\pm, \quad \{D_+, D_-\} = 0.$$

A general bosonic scalar superfield has a component expansion

$$\Phi = \phi + i\theta^+\psi_+ + i\theta^-\psi_- + i\theta^+\theta^-\sigma.$$

Note that the fermionic fields ψ_\pm are components of a spinor which transform under the Lorentz group according to $\psi_\pm \rightarrow e^{\mp\lambda/2}\psi_\pm$, in contrast to the behaviour of the fermionic coordinates. This is consistent because spinor indices can be raised and lowered by reversing signs, though we shall never need to make use of this. We shall write spinor indices explicitly so as to avoid the introduction of Dirac matrices. However, we will adopt the usual notation for the Lorentz-invariant and parity-even inner-product

$$\bar{\chi}\psi = i(\chi_+\psi_- + \psi_+\chi_-),$$

which is real and manifestly symmetric for real fermions.

Now given any simple root system of a Lie superalgebra $\mathcal{R} = \mathcal{B} \cup \mathcal{F}$ we can write down a corresponding integrable Toda theory in terms of a real scalar superfield Φ which takes values in the target space $V = \langle \mathcal{R} \rangle$. The equations of motion are

$$iD_+D_-\Phi = \sum_{j \in \tau} \mu_j \alpha_j \exp \alpha_j \cdot \Phi + i\theta^+\theta^- \sum_{j \in \varepsilon} \mu_j \alpha_j \exp \alpha_j \cdot \Phi \quad (3.1)$$

and they can be derived from the superspace lagrangian density

$$L = \frac{i}{2} D_+\Phi \cdot D_-\Phi + \sum_{j \in \tau} \mu_j \exp \alpha_j \cdot \Phi + i\theta^+\theta^- \sum_{j \in \varepsilon} \mu_j \exp \alpha_j \cdot \Phi. \quad (3.2)$$

For discussions of integrability of these equations see [1–6, 9–13, 16–22, 25]. The quantities μ_i are non-zero parameters with the dimensions of $(mass)$ for $i \in \tau$ and $(mass)^2$ for $i \in \varepsilon$. These parameters can clearly be modified to some extent by shifting the fields Φ , but it is convenient to keep their values arbitrary for the moment. If desired one can introduce a dimensionless coupling β by taking $L \rightarrow L/\beta^2$, and often the Toda equations are written in a way which corresponds to taking $\Phi \rightarrow \beta\Phi$ in addition. This coupling will not be very important for us, since we work entirely classically.

To pass to the component field formulation of the model, we expand the exponential terms and eliminate the auxiliary fields σ in the standard fashion to obtain

$$L = L_{\text{bos}} + L_{\text{ferm}} + L_{\text{int}}, \quad (3.3)$$

where the terms appearing in the component lagrangian are

$$\begin{aligned}
L_{\text{bos}} &= \frac{1}{2} \partial_+ \phi \cdot \partial_- \phi - \sum_{j \in \varepsilon} \mu_j \exp \alpha_j \cdot \phi - \sum_{i, j \in \tau} \frac{1}{2} \mu_i \mu_j (\alpha_i \cdot \alpha_j) \exp(\alpha_i + \alpha_j) \cdot \phi \\
L_{\text{ferm}} &= \frac{i}{2} \psi_+ \cdot \partial_- \psi_+ + \frac{i}{2} \psi_- \cdot \partial_+ \psi_- \\
L_{\text{int}} &= \sum_{j \in \tau} i \mu_j (\alpha_j \cdot \psi_+) (\alpha_j \cdot \psi_-) \exp \alpha_j \cdot \phi .
\end{aligned} \tag{3.4}$$

For future reference we also record the component field equations:

$$\begin{aligned}
\partial_+ \partial_- \phi &= - \sum_{j \in \varepsilon} \mu_j \alpha_j \exp \alpha_j \cdot \phi - \sum_{i, j \in \tau} \frac{1}{2} \mu_i \mu_j (\alpha_i + \alpha_j) (\alpha_i \cdot \alpha_j) \exp(\alpha_i + \alpha_j) \cdot \phi \\
&\quad + \sum_{j \in \tau} i \mu_j \alpha_j (\alpha_j \cdot \psi_+) (\alpha_j \cdot \psi_-) \exp \alpha_j \cdot \phi , \\
\partial_{\mp} \psi_{\pm} &= \mp \sum_{j \in \tau} \mu_j \alpha_j (\alpha_j \cdot \psi_{\mp}) \exp \alpha_j \cdot \phi .
\end{aligned} \tag{3.5}$$

It should be clear either from the superspace description or from the component version written above that the Toda model really does depend on the particular simple root system \mathcal{R} rather than on the superalgebra $\mathcal{G}(\mathcal{R})$. We will see examples below of Toda models with very different properties which are based on inequivalent simple root systems for the same superalgebra.

The bosonic sector, with lagrangian L_{bos} in (3.4), must be integrable by itself, because if the fermions are set to zero at some initial time then they remain zero according to the full set of equations of motion. But it is clear that this bosonic model is not necessarily a Toda theory based on a *simple* Lie algebra. Rather, L_{bos} will generally consist of a sum of several independent bosonic Toda theories, possibly including a number of massless free fields. Each of these independent factors may contribute to the lagrangian with either a positive or a negative sign. It is therefore appropriate at this stage to try to clarify some troublesome points concerning the positivity of actions, kinetic energy, and potential energy.

It is very important to distinguish between the *overall* sign of the lagrangian, which is irrelevant at the classical level, and the *relative* sign between the kinetic and potential terms, which has much more immediate physical significance. This relative sign determines whether a given classical solution is stable, and in particular the sign must be chosen correctly if the theory is to have oscillations with positive $(\text{mass})^2$ about a given stationary point of the potential. This is why we consider a bosonic lagrangian $\frac{1}{2} \partial_+ \phi \cdot \partial_- \phi - U(\phi)$ to be physically sensible when the inner-product in the kinetic term is positive-definite and the potential $U(\phi)$ is bounded below. We shall say that any lagrangian or action of this type is *positive-definite*, or that it has *positive signature*. But from the classical point of view we could equally well change the lagrangian by an overall sign, or equivalently take the inner-product to be negative-definite and the potential to be bounded above. We refer to such a lagrangian or its action as being *negative-definite* or having *negative signature*. If the inner-product is neither positive- nor negative-definite we shall say that the lagrangian has *indefinite signature*.

We must expect L_{bos} in (3.4) to have indefinite signature, in general. The corresponding Toda equations (3.5) will still be integrable (in the sense discussed in [9-25]) no matter what values of the parameters μ_i we choose, but when we pay closer attention to the relative signs between kinetic and potential terms in (3.4), we find that there are some choices of particular physical interest. In fact, we can always ensure that L_{bos} is a combination of bosonic Toda models which are all *individually* positive- or negative-definite, even though their sum may have indefinite signature.

Proposition 2: Consider the Toda theory based on a simple root system \mathcal{R} . The bosonic sector has a lagrangian of the form

$$L_{\text{bos}} = L_+ + L_- + L_0 \quad (3.6)$$

where L_{\pm} are lagrangians for bosonic Toda theories based on simple root systems \mathcal{B}_{\pm} and L_0 is the lagrangian for a number of free massless scalar fields. This sum of lagrangians corresponds to an orthogonal decomposition of the target space $V = V_+ \oplus V_- \oplus V_0$ where $V_{\pm} = \langle \mathcal{B}_{\pm} \rangle$ are the target spaces for L_{\pm} and $V_0 = (\langle \mathcal{B}_+ \rangle \oplus \langle \mathcal{B}_- \rangle)^{\perp}$ is the target space for L_0 . The lagrangians L_{\pm} are positive/negative-definite if we choose $\mu_i \gtrless 0$ for $\alpha_i^2 \gtrless 0$ when $i \in \varepsilon$, and μ_i all with the same sign when $i \in \tau$. Each of the scalar fields in L_0 may contribute with a positive or negative sign.

Proof: These statements can all be checked directly from (3.4) in conjunction with the results of Proposition 1. In particular, part (b) of Proposition 1 ensures the positive/negative-definite nature of L_{\pm} with the choices of signs for μ_i given above.

This gives an essentially complete description of the bosonic sector of any Toda theory based on a superalgebra. Moreover, the relevant semi-simple algebras with root systems \mathcal{B}_{\pm} can be determined very easily for any given \mathcal{R} by using part (b) of Proposition 1 together with the information summarized in Figure 4. Taken together, these facts constitute a simple diagrammatic method for determining the bosonic sector of any Toda theory, and much of what follows in the rest of this paper rests on this idea. Some examples will be given in the following sections.

When we turn to consider the fermions in (3.4) we see that L_{ferm} is a free lagrangian, and that all the remaining non-trivial interactions are contained in L_{int} . We should emphasize that these interactions will link together the various components of L_{bos} described in Proposition 2; it is only when the fermions vanish that these components decouple. We will not attempt to analyze the structure of L_{int} in any generality, except for the following observation.

Proposition 3: Consider the Toda theory based on a simple root system \mathcal{R} . If the inner-product on $V = \langle \mathcal{R} \rangle$ is non-degenerate when restricted to the subspace $\langle \mathcal{F} \rangle$ spanned by the fermionic simple roots, then $V = \langle \mathcal{F} \rangle \oplus \langle \mathcal{F} \rangle^{\perp}$. In this case the only relevant fermionic degrees of freedom are those fields in $\langle \mathcal{F} \rangle$, while those in $\langle \mathcal{F} \rangle^{\perp}$ obey free, massless equations of motion and are decoupled from all other fields.

Proof: First notice that if ω is any vector in $\langle \mathcal{F} \rangle^{\perp}$ then the fermion field $\omega \cdot \psi_{\pm}$ always obeys a free equation of motion according to (3.5). To show that a given field decouples, however, we must also demonstrate that it does not act as a source for other fields by being present in the interaction terms. From the expression for L_{int} , it is clear that the fermion fields which are absent from the interaction terms are precisely those living in the subspace $\langle \mathcal{F} \rangle^{\perp}$. When the inner-product is non-degenerate on $\langle \mathcal{F} \rangle$ we can decompose any ψ as a sum of fields in $\langle \mathcal{F} \rangle$ and $\langle \mathcal{F} \rangle^{\perp}$ in a unique way, and the result follows.

Corollary: If there are no fermionic null roots in \mathcal{R} , the fermions in $\langle \mathcal{F} \rangle^\perp$ always decouple, since the inner-product is positive-definite on $V = \langle \mathcal{R} \rangle$ and so non-degenerate on *any* subspace (this is the situation discussed in [9,17]). An even more special case arises when there are no fermionic roots at all, so $\mathcal{R} = \mathcal{B}$. Then all the fermions are decoupled, L_{bos} is the usual bosonic Toda theory based on \mathcal{B} and is the only non-trivial part of (3.4). In this sense the superspace construction contains the usual bosonic Toda construction [1-4] as a special case.

We now summarize the classification of two important categories of Toda models. It turns out that they share the rather special property that their bosonic sectors correspond to the *entire* bosonic subalgebras $\mathcal{G}(\mathcal{R})_{\text{bos}}$ of $\mathcal{G}(\mathcal{R})$.

Proposition 4: Consider the Toda theory based on a simple root system \mathcal{R} .

- (a) The theory is supersymmetric iff the simple root system is entirely fermionic, $\mathcal{R} = \mathcal{F}$; a complete list is given in Table 1 (following [10,13,15]).
- (b) The theory with real fields has a positive-definite lagrangian iff \mathcal{R} has no null roots; a complete list of such systems involving at least one fermionic root is given in Table 2 (following [32,9]).
- (c) In each of these cases we have $\mathcal{G}(\mathcal{B}_+) \oplus \mathcal{G}(\mathcal{B}_-) = \mathcal{G}(\mathcal{R})_{\text{bos}}$, whereas in general the left-hand side can be a proper subalgebra.

Table 1: Totally fermionic simple root systems

(a) Linearly independent systems for finite-dimensional superalgebras

Superalgebra	Simple root system	Bosonic subalgebra
$A(m, m-1)$	A^* $2m$ nodes	$A_m \oplus A_{m-1} \oplus U(1)$
$B(m-1, m)$	B^* $2m-1$ nodes	$B_{m-1} \oplus C_m$
$B(m, m)$	B^* $2m$ nodes	$B_m \oplus C_m$
$D(m+1, m)$	D^* $2m+1$ nodes	$D_{m+1} \oplus C_m$
$D(m, m)$	D^* $2m$ nodes	$D_m \oplus C_m$
$D(2, 1; \alpha)$	—	$A_1 \oplus A_1 \oplus A_1$

(b) Linearly dependent systems for finite-dimensional superalgebras

Superalgebra	Simple root system	Bosonic subalgebra
$A(m, m)$	A^* $2m+1$ nodes	$A_m \oplus A_m$

(c) Linearly dependent systems for infinite-dimensional superalgebras

Superalgebra	Simple root system	Bosonic subalgebra
$A(m, m)^{(1)}$	\hat{A}^* $2m+2$ nodes	$A_m^{(1)} \oplus A_m^{(1)}$
$A(2m, 2m)^{(4)}$	B^*-B^* $2m+1$ nodes	$A_{2m}^{(2)} \oplus A_{2m}^{(2)}$
$A(2m+1, 2m+1)^{(2)}$	D^*-D^* $2m+3$ nodes	$A_{2m+1}^{(2)} \oplus A_{2m+1}^{(2)}$
$A(2m+1, 2m)^{(2)}$	B^*-D^* $2m+2$ nodes	$A_{2m+1}^{(2)} \oplus A_{2m}^{(2)}$
$B(m, m)^{(1)}$	B^*-D^* $2m+1$ nodes	$B_m^{(1)} \oplus C_m^{(1)}$
$D(m+1, m)^{(1)}$	D^*-D^* $2m+2$ nodes	$D_{m+1}^{(1)} \oplus C_m^{(1)}$
$D(m, m)^{(2)}$	B^*-B^* $2m$ nodes	$D_m^{(2)} \oplus C_m^{(1)}$
$D(2, 1; \alpha)^{(1)}$	—	$A_1^{(1)} \oplus A_1^{(1)} \oplus A_1^{(1)}$

For all entries $m \geq 1$, where we define $D(1, m) = C(m+1)$ and note $D(1, 1) = C(2) = A(1, 0)$. Each of the superalgebras listed has a unique purely-fermionic simple root system which we have expressed in terms of the building blocks introduced in Figure 2. The notation \hat{A}^* stands for a diagram consisting of a circle of grey nodes, and the other symbols indicate diagrams formed by joining building blocks of types B^* and D^* , as discussed in the last section. Dynkin diagrams for the exceptional D families may be found in [37].

Table 2: Simple root systems with positive-definite inner-product

(a) Linearly independent systems for finite-dimensional superalgebras

Superalgebra	Simple root system	Bosonic subalgebra
$B(0, m)$	B_* m nodes	C_m

(b) Linearly dependent systems for infinite-dimensional superalgebras

Superalgebra	Simple root system	Bosonic subalgebra
$B(0, m)^{(1)}$	B_*-C $m+1$ nodes	$C_m^{(1)}$
$A(2m+1, 0)^{(2)}$	B_*-D $m+2$ nodes	$A_{2m+1}^{(2)}$
$A(2m, 0)^{(4)}$	B_*-B $m+1$ nodes	$A_{2m}^{(2)}$
$C(m+1)^{(2)}$	B_*-B_* $m+1$ nodes	$C_m^{(1)}$

For all entries $m \geq 1$, and the building blocks signifying the type of diagram include B_* introduced in Figure 3. (The bosonic sectors given in Table 1 of [9] are incorrect in some instances; this was also noted in [24].)

We explained above that the overall sign of a lagrangian is essentially unimportant for a classical theory. But the situation is rather different at the quantum level: the definition and interpretation of a quantum theory becomes considerably more subtle if the lagrangian is not positive-definite. Now on the one hand we have grown accustomed in recent years to the idea that indefinite or complex lagrangians can be useful quantum-mechanically in the description of non-unitary models, or even of unitary models if some projection onto a subspace of physical states can be implemented [38,39,40]. Having said that, it is also natural to retain a particular interest in those actions which are positive-definite—like those listed in Table 2—which can be quantized in an entirely straightforward way. In this paper we will certainly keep an open mind regarding the likely importance of theories with indefinite signature; but we will also single out for special attention those theories which are positive-definite.

4. Examples with extended conformal symmetry

Toda theories provide a valuable source of examples of extended conformal symmetries, in which the Virasoro algebra is embedded within some larger \mathcal{W} -algebra. In fact the Toda construction, or the related procedure of Hamiltonian reduction, provides one of the most systematic ways of constructing \mathcal{W} -algebras, and may perhaps lead to some eventual classification of them. The \mathcal{W} -algebras arising from bosonic Toda theories have been investigated from a variety of different points of view. The \mathcal{W} -algebras arising from

superalgebra models are much less well studied. In this context our results concerning the structure of a general Toda model can provide some basic information on the bosonic sectors of these new extended conformal symmetries.

Let us now briefly recall how conformal symmetry arises. The following statements are well-known but it is useful to collect them in a proposition for completeness and for future reference.

Proposition 5: Given a simple root system \mathcal{R} which is linearly independent, the corresponding Toda model is invariant under conformal transformations $x^\pm \rightarrow y^\pm(x^\pm)$ acting on the fields according to

$$\begin{aligned}\phi(x^+, x^-) &\rightarrow \phi(y^+, y^-) + \rho \log(\partial_+ y^+ \partial_- y^-) \\ \psi_\pm(x^+, x^-) &\rightarrow (\partial_\pm y^\pm)^{1/2} \psi_\pm(y^+, y^-)\end{aligned}\tag{4.1}$$

where the vector ρ is defined by the conditions

$$\rho \cdot \alpha_i = \begin{cases} 1 & i \in \varepsilon \\ 1/2 & i \in \tau \end{cases}$$

Linear independence of \mathcal{R} (or $n = r$) ensures that ρ exists; this holds for all finite-dimensional Lie superalgebras, except for the family $A(m, m)$ (which is unusual in having $n = r+1$). If $\mathcal{R} = \mathcal{F}$, so that the model is also supersymmetric, then conformal invariance is extended to superconformal invariance. A complete list of superconformal models is therefore given in Table 1, section (a).

Proof: These statements can be verified directly from (3.4) or (3.5); for details see eg. [13].

A consequence of \mathcal{R} being linearly independent is that we can always re-define the Toda fields by additive constants so as to make $|\mu_i| = 1$. The fact that these mass parameters can essentially be eliminated is a manifestation of conformal invariance. The inhomogeneous pieces in the transformation of the bosons in (4.1) give rise to a classical central charge $12\rho^2$ but this is generally modified quantum-mechanically—see [2,13] and references given there.

To illustrate the results of the previous section, we list in Table 3 all linearly independent simple root systems of rank 1 or 2, together with a selection of root systems of rank 3. In each case we specify \mathcal{R} by its Dynkin diagram and give the corresponding bosonic systems \mathcal{B}_\pm as found by our diagrammatic method. The number of free scalar fields in the bosonic sector of the Toda theory is indicated by the dimension of V_0 , and in all these examples they contribute with a negative sign. Simple root systems of higher rank can be considered in an exactly similar fashion.

The examples labeled (1)-(6) are particularly instructive and so we shall examine them more closely. We adopt the notation $\Phi_i = \alpha_i \cdot \Phi$ for superfields and their components. It will be convenient to denote the lagrangians for a free massless boson ϕ and fermion ψ by

$$L_{\text{free}}(\phi) = \partial_+ \phi \partial_- \phi, \quad L_{\text{free}}(\psi) = i\psi_+ \partial_- \psi_+ + i\psi_- \partial_+ \psi_- ,\tag{4.2}$$

respectively. The overall normalizations for various lagrangians appearing below will not always be standard, but we choose them in order to eliminate as many awkward numerical factors as possible.

Table 3: Some low-rank linearly-independent simple root systems

	Superalgebra	\mathcal{R}	\mathcal{B}_+	\mathcal{B}_-	$\dim V_0$
(1)	A_1			—	0
(2)	$B(0,1)$			—	0
(3)	$B(0,2)$			—	0
(4)	$A(1,0)$			—	1
(5)	$A(1,0)$			—	1
	$B(1,1)$			—	1
	$B(1,1)$				0
	$A(2,0)$			—	1
	$B(1,2)$			—	1
(6)	$B(2,1)$				0
	$C(3)$			—	1
	$D(2,1)$				0

(1) A_1 : the simplest possible Lie algebra produces the prototype of all Toda models—the Liouville theory. This consists of a single bosonic field ϕ with lagrangian

$$L_{\text{exp}}(\phi) = \partial_+ \phi \partial_- \phi - \exp 2\phi . \quad (4.3)$$

In the superspace approach, this actually comes accompanied by a free fermion ψ so that

$$L_{\text{bos}} = L_{\text{exp}}(\phi) , \quad L_{\text{ferm}} = L_{\text{free}}(\psi) . \quad (4.4)$$

As for any bosonic algebra, however, the fermionic degrees of freedom decouple completely.

(2) $B(0,1)$: the simplest possible superalgebra with a single fermionic simple root. On choosing a suitable normalization for the fields, the expressions for L_{bos} and L_{ferm} given in (4.4) above are unchanged. But the fermion is no longer free; instead it is coupled to the Liouville field through the interaction term

$$L_{\text{int}} = \bar{\psi} \psi \exp \phi . \quad (4.5)$$

This is the supersymmetric Liouville theory (see eg. [11] and references given there).

(3) $B(0,2)$: \mathcal{R} has $\tau = \{1\}$ and Cartan matrix

$$a_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} .$$

The full set of equations of motion is

$$\begin{aligned}\partial_+ \partial_- \phi_1 &= -\exp 2\phi_1 + \exp \phi_2 + \frac{1}{2} \bar{\psi}_1 \psi_1 \exp \phi_1 , & \partial_{\pm} \psi_{1\mp} &= \pm \psi_{1\pm} \exp \phi_1 , \\ \partial_+ \partial_- \phi_2 &= \exp 2\phi_1 - 2 \exp \phi_2 - \frac{1}{2} \bar{\psi}_1 \psi_1 \exp \phi_1 , & \partial_{\pm} \psi_{2\mp} &= \mp \psi_{1\pm} \exp \phi_1 .\end{aligned}$$

We can see directly from these equations that the bosonic sector is a C_2 Toda theory with fields $2\phi_1$ and ϕ_2 , confirming the information given in Table 3. It is also evident that the only relevant fermionic field is ψ_1 , since the combination $\psi_1 + \psi_2$ decouples, in keeping with Proposition 3. Following on from example (2), this is the next member of a series of conformally-invariant models based on the root systems for $B(0, n)$ [17]—see Table 2 (a). For $n > 1$ the models are not supersymmetric; they consist of a C_n Toda theory, with the field $2\phi_1$ corresponding to the long root, and ϕ_1 coupled to a single fermion ψ_1 through an interaction of type (4.5).

(4) $A(1,0) = C(2)$: \mathcal{R} has $\tau = \{1, 2\}$ and Cartan matrix

$$a_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The equations of motion now read

$$\begin{aligned}\partial_+ \partial_- \phi_1 &= -\exp(\phi_1 + \phi_2) + \frac{1}{2} \bar{\psi}_2 \psi_2 \exp \phi_2 , & \partial_{\pm} \psi_{1\mp} &= \pm \psi_{2\pm} \exp \phi_2 , \\ \partial_+ \partial_- \phi_2 &= -\exp(\phi_1 + \phi_2) + \frac{1}{2} \bar{\psi}_1 \psi_1 \exp \phi_1 , & \partial_{\pm} \psi_{2\mp} &= \pm \psi_{1\pm} \exp \phi_1 .\end{aligned}$$

Since the root system is entirely fermionic, this model is supersymmetric. The bosonic sector consists of an A_1 Toda, or Liouville theory, together with a scalar field. More precisely, if we introduce the fields

$$2u = \phi_1 + \phi_2 \quad \text{and} \quad 2v = \phi_1 - \phi_2 \tag{4.6}$$

then the bosonic sector is

$$L_{\text{bos}} = L_+ + L_0 = L_{\text{exp}}(u) - L_{\text{free}}(v) . \tag{4.7}$$

Similarly, if we adopt corresponding redefinitions for the fermions

$$2\chi = \psi_1 + \psi_2 \quad \text{and} \quad 2\eta = \psi_1 - \psi_2 \tag{4.8}$$

then we have

$$L_{\text{ferm}} = L_{\text{free}}(\chi) - L_{\text{free}}(\eta) . \tag{4.9}$$

We can complete the description of the model in terms of these new fields by giving the interaction term

$$L_{\text{int}} = (\bar{\chi}\chi + \bar{\eta}\eta) \exp u \cosh v + 2\bar{\chi}\eta \exp u \sinh v . \tag{4.10}$$

This example is known to be closely related to the $N = 2$ super Liouville theory; we will return to it briefly in section 7.

(5) $A(1,0)=C(2)$: \mathcal{R} has $\tau = \{2\}$ and Cartan matrix

$$a_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} .$$

The equations of motion are

$$\begin{aligned} \partial_+ \partial_- \phi_1 &= -2 \exp \phi_1 - \frac{1}{2} \bar{\psi}_2 \psi_2 \exp \phi_2 , & \partial_{\pm} \psi_{1\mp} &= \mp \psi_{2\pm} \exp \phi_2 , \\ \partial_+ \partial_- \phi_2 &= \exp \phi_1 , & \partial_{\pm} \psi_{2\mp} &= 0 . \end{aligned}$$

The bosonic sector is again based on $A_1 \oplus U(1)$ where now ϕ_1 is the Liouville field and $\phi_1 + 2\phi_2$ is the scalar, but apart from this the equations of motion make clear that this model bares little resemblance to the previous example based on the same algebra, eg. it is not supersymmetric. This illustrates the fact that the properties of the Toda theory depend fundamentally on the root system rather than on the algebra which it defines. In general there is no reason to expect even that the bosonic sectors should coincide. Notice also that neither of the fermions are decoupled in this example, despite the fact that ψ_2 obeys a free equation of motion.

(6) $C(3)$: \mathcal{R} has $\tau = \{1, 2\}$ and Cartan matrix

$$a_{ij} = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix} .$$

The bosonic sector corresponds to $C_2 \oplus U(1)$. As in example (4), it is useful to introduce the combinations of fields (4.6) and (4.8). The expression for the bosonic lagrangian is then similar to (4.7) except that L_+ is now a lagrangian for a C_2 Toda model with fields u and ϕ_3 while v is still the free scalar which contributes with opposite sign. Proposition 3 states that ψ_3 decouples, leaving χ and η as the only important fermionic degrees of freedom, and (4.9) is unchanged. The interaction term (4.10) is also unchanged. This analysis can be generalized immediately to any of the algebras $C(m)$ with root systems of type D_* . The lagrangian L_+ becomes a C_{m-1} Toda model in general, but there are only ever two relevant fermions which interact with the bosons in just the way we have described.

5. Examples without conformal symmetry

For bosonic Toda theories there is a clear-cut distinction between those based on finite-dimensional algebras, which are conformally invariant, and those based on affine algebras, which are massive. In the last section we saw that Toda theories based on finite-dimensional superalgebras are also conformally invariant (with the exception of the $A(m, m)$ series) just as in the bosonic case. When we come to Toda models based on affine superalgebras, however, we find a whole range of new possibilities. Some models resemble their bosonic siblings in that they consist entirely of massive degrees of freedom. But others contain mixtures of massless and massive modes, and some may have no massive degrees of freedom at all in their bosonic sectors.

Table 4: Some low-rank linearly-dependent simple root systems

	Superalgebra	\mathcal{R}	\mathcal{B}_+	\mathcal{B}_-	$\dim V_0$
(1)	$A_1^{(1)}$				0
(2)	$A(1,0)^{(2)}$				0
	$A(2,0)^{(4)}$				0
	$B(0,1)^{(1)}$				0
	$A(2,2)^{(4)}$				0
	$D(2,1)^{(2)}$				0
	$A(2,1)^{(2)}$				0
(3)	$B(1,1)^{(1)}$				0
	$A(2,1)^{(2)}$				1
	$A(1,0)^{(1)}$				1
(4)	$A(1,1)^{(1)}$				0
	$A(1,1)^{(1)}$				0
	$A(2,0)^{(1)}$				1
	$A(3,1)^{(2)}$				1
(5)	$D(2,1)^{(1)}$				0

We can readily see how these various possibilities arise by applying our diagrammatic method to determine the bosonic sectors of a number of representative examples, with results summarized in Table 4. It is generally straightforward to combine the diagrams in Figures 2, 3 and 4 in order to extract the necessary information. The only cases in which a little extra thought may be required are those root systems with a very small number of nodes so that the tails of the affine diagrams are not well-separated; we have included the least obvious examples in the table.

To amplify our previous remarks, we note that some of these models are purely massive, including those labeled (1)-(5); some contain mixtures of massive and massless degrees of freedom (the most trivial possible conformal theory being a free massless field); and one theory, based on the second choice of simple roots for $A(1, 1)^{(1)}$, has a conformally-invariant bosonic sector. Many more examples of each kind can be found by looking at higher-rank diagrams. The cases (1)-(5) turn out to be of special interest, so we now consider them in more detail.

We mentioned in the last section that for \mathcal{R} linearly independent, $r = n$, we can choose the values of $|\mu_i|$ as we please by making constant shifts in the fields. In this section we are taking \mathcal{R} to be linearly dependent. All but one family of simple roots of this type have $n = r+1$, in common with bosonic affine algebras. This means that they obey exactly one linear relation and so, while we may be free to alter *ratios* of the parameters μ_i , there is always one surviving overall mass scale m which cannot be eliminated. The exceptions, with $n = r+2$, are simple root systems for the algebras $A(m, m)^{(1)}$. In these cases there are two independent mass parameters m and m' , or equivalently a single mass parameter m and an additional dimensionless ratio $\gamma = m/m'$ (in addition to the possibility of introducing the usual kind of Toda coupling β which we are omitting throughout). In all the examples below μ_i will be chosen to be a zero-eigenvector of the Cartan matrix, ensuring that the minimum of the bosonic potential occurs where the fields vanish. As with the conformal examples, we adopt overall normalizations so as to fix various numerical constants in a simple way.

(1) $A_1^{(1)}$: the simplest affine Lie algebra, with Cartan matrix

$$a_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} .$$

Note that the rows are proportional to one another, implying a linear relation between the superfields $\Phi_1 + \Phi_2 = 0$. This results is the prototype of all affine Toda theories—the sinh-Gordon theory. There is a single independent bosonic field ϕ of mass m governed by the lagrangian

$$\begin{aligned} L_{\text{sinh}}(\phi; m) &= \partial_+ \phi \partial_- \phi - m^2 \sinh^2 \phi \\ &= \partial_+ \phi \partial_- \phi - \frac{1}{2} m^2 (\cosh 2\phi - 1) , \end{aligned} \tag{5.1}$$

where we have added a constant to make the minimum value of the potential zero. As with the Liouville theory, this is accompanied in the superspace construction by a free massless fermion ψ so that

$$L_{\text{bos}} = L_{\text{sinh}}(\phi; m) , \quad L_{\text{ferm}} = L_{\text{free}}(\psi) , \tag{5.2}$$

but the boson and fermion do not interact with one another. The relationship with the sine-Gordon theory will be discussed in section 7.

(2) $A(1, 0)^{(2)} = C(2)^{(2)}$: defined by the same Cartan matrix as above, but now the roots are fermionic and the theory is supersymmetric. With a suitable normalization for the fields, the terms L_{bos} and L_{ferm} are unchanged, but there is an additional interaction term

$$L_{\text{int}} = m \bar{\psi} \psi \cosh \phi \tag{5.3}$$

which gives the fermion a mass m equal to that of the boson. This is the supersymmetric sinh-Gordon theory.

(3) $B(1, 1)^{(1)}$: \mathcal{R} has $\tau = \{1, 2, 3\}$ and Cartan matrix

$$a_{ij} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 2 \\ -1 & 2 & 0 \end{pmatrix} .$$

There is a single linear relation amongst the simple roots, and consequently the superfields obey $2\Phi_1 + \Phi_2 + \Phi_3 = 0$. The bosonic sector consists of a pair of $A_1^{(1)}$ Toda models, or sinh-Gordon theories, which contribute with opposite signs. To see this explicitly we introduce the independent fields

$$u = -\phi_1 = \phi_1 + \phi_2 + \phi_3 , \quad 2v = \phi_1 + \phi_2 = -(\phi_1 + \phi_3) ,$$

and in terms of these we find

$$L_{\text{bos}} = L_+ + L_- = L_{\text{sinh}}(u; 2m) - 4L_{\text{sinh}}(v; m) . \quad (5.4)$$

Note that the masses of these two theories are different and also that the relative coefficient between their lagrangians cannot be changed. If we introduce the corresponding fermionic fields

$$\chi = -\psi_1 = \psi_1 + \psi_2 + \psi_3 , \quad 2\eta = \psi_1 + \psi_2 = -(\psi_1 + \psi_3) ,$$

then we find similarly

$$L_{\text{ferm}} = L_{\text{free}}(\chi) - 4L_{\text{free}}(\eta) . \quad (5.5)$$

The interaction term is also easily found to be

$$L_{\text{int}} = m \{ \bar{\chi}\chi \exp(-u) + (\bar{\chi}\chi + 4\bar{\eta}\eta) \exp u \cosh 2v + 4\bar{\chi}\eta \exp u \sinh 2v \} . \quad (5.6)$$

(4) $A(1, 1)^{(1)}$: \mathcal{R} is defined by $\tau = \{1, 2, 3, 4\}$ and the Cartan matrix

$$a_{ij} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} .$$

As mentioned earlier, there are two linear relations amongst simple roots for these algebras, and in this case we have $\Phi_1 + \Phi_3 = \Phi_2 + \Phi_4 = 0$. Once again the bosonic sector consists of a pair of sinh-Gordon theories which contribute with opposite overall signs. We introduce

$$\begin{aligned} 2u = \phi_1 + \phi_2 = -(\phi_3 + \phi_4) , & \quad 2\chi = \psi_1 + \psi_2 = -(\psi_3 + \psi_4) , \\ 2v = \phi_1 + \phi_4 = -(\phi_2 + \phi_3) , & \quad 2\eta = \psi_1 + \psi_4 = -(\psi_2 + \psi_3) , \end{aligned}$$

and then the bosonic sector is given by

$$L_{\text{bos}} = L_+ + L_- = L_{\text{sinh}}(u; m) - L_{\text{sinh}}(v; m) , \quad (5.7)$$

while in the fermionic sector

$$L_{\text{ferm}} = L_{\text{free}}(\chi) - L_{\text{free}}(\eta) . \quad (5.8)$$

Lastly, we find the interaction term

$$\begin{aligned}
L_{\text{int}} &= \frac{1}{2}m(\bar{\chi}\chi + \bar{\eta}\eta) \{ \gamma \cosh(u+v) + \gamma^{-1} \cosh(u-v) \} \\
&\quad + m\bar{\chi}\eta \{ \gamma \cosh(u+v) - \gamma^{-1} \cosh(u-v) \} \\
&= m(\bar{\chi}\chi + \bar{\eta}\eta) \cosh u \cosh v + 2m\bar{\chi}\eta \sinh u \sinh v \quad \text{when } \gamma = 1 .
\end{aligned} \tag{5.9}$$

Notice that the masses of the sinh-Gordon theories appearing in L_{\pm} are equal, irrespective of the value of the dimensionless parameter γ . The case $\gamma = 1$ is given explicitly in preparation for some observations to follow in Section 7.

(5) $D(2,1)^{(1)}$: \mathcal{R} defined by $\tau = \{1, 2, 3, 4\}$ and Cartan matrix

$$a_{ij} = \begin{pmatrix} 0 & 2 & -1 & -1 \\ 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 2 \\ -1 & -1 & 2 & 0 \end{pmatrix} .$$

Here there is one linear relation $\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 = 0$. In this case the bosonic sector consists of three sinh-Gordon theories, one contributing positively, the others negatively. We introduce the independent fields

$$\begin{aligned}
2u &= \phi_1 + \phi_2 = -(\phi_3 + \phi_4) , & 2\chi &= \psi_1 + \psi_2 = -(\psi_3 + \psi_4) , \\
2v_1 &= \phi_1 + \phi_3 = -(\phi_2 + \phi_4) , & 2\eta_1 &= \psi_1 + \psi_3 = -(\psi_2 + \psi_4) , \\
2v_2 &= \phi_1 + \phi_4 = -(\phi_2 + \phi_3) , & 2\eta_2 &= \psi_1 + \psi_4 = -(\psi_2 + \psi_3) .
\end{aligned}$$

Then the bosonic sector is

$$L_{\text{bos}} = L_+ + L_- = L_{\text{sinh}}(u; 2m) - 2L_{\text{sinh}}(v_1; m) - 2L_{\text{sinh}}(v_2; m) , \tag{5.10}$$

and in the fermionic sector

$$L_{\text{ferm}} = L_{\text{free}}(\psi) - 2L_{\text{free}}(\eta_1) - 2L_{\text{free}}(\eta_2) . \tag{5.11}$$

A short calculation reveals the interaction term

$$\begin{aligned}
L_{\text{int}} &= m(\bar{\chi}\chi + \bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2) \{ \exp u \cosh(v_1 + v_2) + \exp(-u) \cosh(v_1 - v_2) \} \\
&\quad + 2m\bar{\eta}_1\eta_2 \{ \exp u \cosh(v_1 + v_2) - \exp(-u) \cosh(v_1 - v_2) \} \\
&\quad + 2m\bar{\chi}\eta_1 \{ \exp u \sinh(v_1 + v_2) - \exp(-u) \sinh(v_1 - v_2) \} \\
&\quad + 2m\bar{\chi}\eta_2 \{ \exp u \sinh(v_1 + v_2) + \exp(-u) \sinh(v_1 - v_2) \} .
\end{aligned} \tag{5.12}$$

If we set $v_1 = v_2 = v$ and $\eta_1 = \eta_2 = \eta$ then we recover example (3) above. This is because the simple root systems involved are related by the folding operation discussed for superalgebras in [36].

Given everything we have said so far, it is natural to ask which Toda theories based on superalgebras are purely massive.

Proposition 6: A Toda theory is purely massive iff it is based on a simple root system \mathcal{R} whose Dynkin diagram belongs to one of the following classes:

- (a) Affine diagrams with no grey nodes—in addition to the usual bosonic affine root systems we have precisely the cases listed in Table 2, section (b).
- (b) Affine diagrams which are entirely fermionic, ie. which give massive supersymmetric theories—precisely the cases listed in Table 1, section (c).

Proof: In order to get a purely massive bosonic theory there must be a linear relation amongst the roots in both \mathcal{B}_+ and \mathcal{B}_- , so that they both represent affine bosonic algebras; this means the numbers of nodes in these diagrams, denoted by n , must exceed the dimensions of $V_{\pm} = \langle \mathcal{B}_{\pm} \rangle$. But in addition \mathcal{B}_+ and \mathcal{B}_- must together span V to ensure that V_0 is trivial and so exclude massless scalars. Let us consider first those diagrams with $n = r + 1$, which omits just the root systems of $A(m, m)^{(1)}$. We have one linear relation on \mathcal{R} and hence at least one on $\mathcal{B}_+ \cup \mathcal{B}_-$. If either of \mathcal{B}_{\pm} is empty, then this single linear relation can suffice to make the remaining bosonic root system affine—this gives rise to the class of allowed cases in (a). If neither of \mathcal{B}_{\pm} is trivial, however, then we need one extra linear relation to ensure that both represent bosonic affine root systems. For all allowed Dynkin diagrams, it is easy to see that the number of bonds never exceeds the number of nodes by more than one. Using this in conjunction with the definitions of \mathcal{B}_{\pm} , we see that $n(\mathcal{B}_+) + n(\mathcal{B}_-) \leq n(\mathcal{R}) + 1$, with equality only when $\mathcal{R} = \mathcal{F}$ is purely fermionic. This gives rise to almost all the examples in class (b). Finally, we must consider root systems for $A(m, m)^{(1)}$ with $n = r + 2$. But these diagrams are all circular, which means that the bound used above can be improved to $n(\mathcal{B}_+) + n(\mathcal{B}_-) \leq n(\mathcal{R})$, again with equality only for the purely fermionic case. This allows just the last entry in Table 2, completing the proof.

The examples (1)-(5) from Table 4 all belong to one or other of the classes in Proposition 6. We should emphasize that for class (b)—supersymmetric models—the lagrangians are generally of indefinite signature. Nevertheless, from what we have said in Section 3, the masses of the bosons are perfectly well-defined, since the relative sign between the potential and kinetic energy terms is always physically sensible (provided we choose the parameters μ_i as indicated in Proposition 2). Furthermore, the lists of bosonic sectors given in Tables 1 and 2 tell us immediately the bosonic masses of any of the superalgebra theories in Proposition 6, in terms of the known results for bosonic Toda models [3,8,27]. This provides a proof of the mass formulas for the particular series $A(m, m)^{(1)}$, $B(m, m)^{(1)}$, and $D(m+1, m)^{(1)}$ which were conjectured in [25] on the basis of computer calculations.

For the supersymmetric examples in class (b) of Proposition 6, the fermions must be degenerate in mass with the bosons. Whether these models can be associated with ‘pseudo-unitary’ quantum S-matrices (see eg. [41,40]) as suggested by perturbative calculations [25] is a fascinating issue, but not one that we shall attempt to pursue here. By contrast, the examples in class (a) have conventional positive-definite lagrangians and so they can be quantized straightforwardly. Since they are not supersymmetric, the bosonic and fermionic masses will be different. Some relevant S-matrices are discussed in [24].

6. Conformal curiosities

We stated in Proposition 5 that the existence of a conformal symmetry (4.1) relies on \mathcal{R} being linearly independent. But there are numerous Toda models for which \mathcal{R} is linearly dependent and yet for which the bosonic sector L_{bos} is conformally invariant when taken in isolation. Perhaps the most striking examples are those based on the finite-dimensional algebras $A(m, m)$ whose special properties were noted in section 4, though we have seen similar behaviour for several other models based on infinite-dimensional superalgebras in the last section. In all these cases we must conclude that the interactions with the fermions break the conformal invariance present in the bosonic sector. It is interesting to consider whether the conformal symmetry might be extended in some more subtle way so as to include the fermions.

Keeping to our earlier notation, we write $\phi_j = \alpha_j \cdot \phi$ and $\psi_j = \alpha_j \cdot \psi$, and we consider the transformations

$$\phi_j \rightarrow \phi_j + h_j \log(\partial_+ y^+ \partial_- y^-), \quad \psi_{j\pm} \rightarrow (\partial_{\pm} y^{\pm})^{1-h_j} \psi_{j\pm}, \quad (6.1)$$

where h_j may be a different real number for each fermionic root α_j . The usual choice, corresponding to (4.1), is $h_j = 1/2$. It is easy to see from (3.4) that the relationship between the conformal weights of the bosons and fermions in (6.1) automatically renders the terms in L_{int} invariant. The problem is that it is much more difficult to achieve invariance of the other terms—particularly since a linear relation amongst the simple roots imposes a corresponding relationship amongst the fields and hence amongst the possible values of the numbers h_j . There are, however, two families of examples in which exotic conformal symmetries of this type exist.

First we consider a Toda model based on $A(m, m)$, \mathcal{R} being of type A^* with $2m-1$ nodes. Examination of the Cartan matrix reveals that the linear relation involves only the odd-numbered roots: $\sum_j \alpha_{2j+1} = 0$. Now it is easy to see that in this case (3.4) is invariant if we choose $h_j = 0$ for j odd and $h_j = 1$ for j even, and that this is consistent with the linear relation amongst the fields. (Here ‘odd’ and ‘even’ refer to the values of the integer labels, not to the grading of the roots, which are all fermionic!) This was pointed out in [15] for the simplest case based on $A(1,1)$. The second set of examples is based on $A(m, m-1)$, again with \mathcal{R} of type A^* but this time with $2m$ nodes. Now the simple roots are actually linearly independent, and we have already seen that this model is invariant under the standard conformal transformations with $h_j = 1/2$. But in fact it is invariant under *any* assignment of conformal weights of the type

$$h_j = s \text{ for } j \text{ odd}, \quad h_j = 1-s \text{ for } j \text{ even}, \quad (6.2)$$

where s is any real number.

The existence of these rather strange conformal symmetries can be traced to a specific feature of the simple root systems of type A^* . They can be divided into disjoint sets $\mathcal{F}_0 = \{\alpha_j \in \mathcal{F} \mid j \text{ even}\}$ and $\mathcal{F}_1 = \{\alpha_j \in \mathcal{F} \mid j \text{ odd}\}$ and on each of these individually the inner-product vanishes, so we obtain a non-zero result only when we choose one vector from each set. For this to hold it is obviously necessary that all the simple roots are null, but this is by no means sufficient, and it is not difficult to see that the A^* root systems

are the unique ones having this property. The reason it is important is that the conformal weights of a fermion are usually fixed to be $1/2$ by its kinetic terms, but when we have inner-products of this peculiar type a fermion field never appears paired with itself in the kinetic term, because of the degeneracy.

These remarks strongly suggest that non-standard conformal symmetries (6.1) with $h_j \neq 1/2$ occur only for the models that we have just discussed based on the A^* root systems. Although there are many other Toda models based on linearly-dependent simple root systems with conformally-invariant bosonic sectors, it seems likely that for these the conformal symmetry is irrevocably broken by the addition of fermions. This is similar in many ways to some recent results concerning the incompatibility of $(1,0)$ supersymmetry and extended conformal invariance in Toda models [42].

The emergence of fermions with integer spin is reminiscent of ghosts in a topological field theory. Indeed, the special feature of the A^* root system—that it can be divided into the two disjoint degenerate sets \mathcal{F}_0 and \mathcal{F}_1 —is also known to be linked to the existence of $N = 2$ supersymmetry in certain versions of the $A(m, m-1)$ models [13,14,15]. It is natural to expect that the family of conformal symmetries which exists between $s = 1/2$ and $s = 0$ or 1 should correspond to the usual twisting of an $N = 2$ model into a topological theory. The fact that the conformal symmetry of the $A(m, m)$ model exists only for $s = 0$ means that it does not seem to be immediately related to a topological twisting; nevertheless it would be interesting to investigate whether it too might have a topological character. Topological properties of massive Toda theories have already been investigated in [26].

7. Twisted reality conditions

We now discuss one last ingredient which is important in the taxonomy of Toda models. We have assumed so far that the fields appearing in the Toda equations are real. In fact the integrability of the equations is unaffected if we take all the fields to be complex in (3.1), so there exist complex versions of all the Toda theories we have been considering. A related possibility which is more interesting in many respects is to impose non-standard or *twisted* reality conditions on the fields, corresponding to a reflection symmetry of the simple root system \mathcal{R} . This was first pointed out for abelian Toda models in [14] and applied to the study of $N = 2$ supersymmetry in [14,15]; the construction has recently been related to Hamiltonian reduction and extended to the case of non-abelian Toda models in [19].

In the bosonic case, a symmetry of \mathcal{R} is equivalent to a symmetry of the Dynkin diagram. For superalgebras, however, we encounter once again the problem that the diagram may have more symmetry than the Cartan matrix, because of the now familiar issue of signs. With the approach we have developed in this paper, we can extend the results of [14], clarifying the effect on the bosonic sector.

Proposition 7: Let σ be a permutation of order two acting on a set of simple roots \mathcal{R} .

(a) We can impose reality conditions

$$\alpha_i \cdot \Phi^* = \alpha_{\sigma(i)} \cdot \Phi \quad (7.1)$$

consistent with integrability of the Toda equations based on \mathcal{R} provided:

$$a_{\sigma(i)\sigma(j)} = a_{ij} \quad \text{and} \quad \mu_{\sigma(i)} = \mu_i \quad (7.2)$$

The first condition means that σ can be regarded as a reflection symmetry of the Dynkin diagram *even after attaching signs* as described in Section 2. The second condition is automatically satisfied if we have chosen $|\mu_i| = 1$ when $n = r$, or if we have chosen μ_i to coincide with the unique null eigenvector of the Cartan matrix when $n = r+1$. (We have assumed μ_i real, but there is actually no loss of generality in doing this.)

(b) The permutation σ acts in an obvious way as a symmetry of the associated bosonic root systems \mathcal{B}_\pm . The bosonic sector of the theory defined by (7.1) therefore consists of bosonic Toda theories, as explained in Proposition 2, but each with twisted reality conditions corresponding to these symmetries of \mathcal{B}_\pm .

(c) Taking standard reality conditions, $\sigma = 1$, or twisted reality conditions, $\sigma \neq 1$, does not affect whether the theory is conformally-invariant (under (4.1)), or whether it is massive, in which case the classical mass spectrum is unchanged.

Proof: The results all follow easily from (3.1), (3.5) and the definition of \mathcal{B}_\pm . Notice that the conformal transformations in (4.1) only change the real parts of the fields, no matter what reality conditions are chosen. The mass spectrum is obtained by linearizing the equations of motion (3.5) and so it is clear that this is unchanged.

The imposition of twisted reality conditions amounts to taking certain components of the Toda field to be imaginary rather than real, but in a way which preserves the reality of the lagrangian. This can be thought of as a kind of generalization of the simple behaviour of a free massless scalar field, for which

$$\phi = i\rho \quad \Rightarrow \quad L_{\text{free}}(\phi) = -L_{\text{free}}(\rho) . \quad (7.3)$$

All bosonic Toda theories have lagrangians of definite signature, so changing the reality conditions of the fields must always lead to a lagrangian of indefinite signature, except perhaps when there is only one independent field, as for the free scalar above. The behaviour (7.3) cannot be generalized to the Liouville theory, because there is no non-trivial symmetry of the Dynkin diagram for A_1 , but it can be generalized to its affine counterpart, the sinh-Gordon theory, based on the algebra $A_1^{(1)}$.

We discussed the sinh-Gordon theory as example (1) from Table 4, and we saw that the model consists of a single bosonic field (since the accompanying free fermion can essentially be ignored) with lagrangian

$$\begin{aligned} L_{\text{sinh}}(\phi; m) &= \partial_+ \phi \partial_- \phi - m^2 \sinh^2 \phi \\ &= \partial_+ \phi \partial_- \phi - \frac{1}{2} m^2 (\cosh 2\phi - 1) . \end{aligned} \quad (7.4)$$

There is clearly a symmetry σ of the Dynkin diagram which exchanges its two nodes and corresponds to reflection in a vertical axis through the figure drawn in Table 4. The imposition of reality conditions (7.1) corresponding to this symmetry means that we must set $\phi = i\rho$ where ρ is real. But

$$\phi = i\rho \quad \Rightarrow \quad L_{\text{sinh}}(\phi; m) = -L_{\text{sin}}(\rho; m) , \quad (7.5)$$

where we introduce the well-known sine-Gordon lagrangian

$$\begin{aligned} L_{\text{sin}}(\rho; m) &= \partial_+ \rho \partial_- \rho - m^2 \sin^2 \rho \\ &= \partial_+ \rho \partial_- \rho + \frac{1}{2} m^2 (\cos 2\rho - 1) . \end{aligned} \quad (7.6)$$

Twisted reality conditions are therefore important in understanding how the sine-Gordon theory fits into the general framework of the Toda construction. We repeat that this is the only bosonic example for which a lagrangian of definite signature (albeit negative) emerges when twisted reality conditions are used; all other bosonic theories acquire indefinite signature.

These observations are easily extended to the supersymmetric sinh-Gordon theory, example (2) of Table 4. Again the symmetry corresponds to reflection in vertical axes through the diagrams in the Table. When (7.1) is applied to the fermions we must take $\psi = i\lambda$ with λ real. But then

$$\psi = i\lambda \quad \Rightarrow \quad L_{\text{free}}(\psi) = -L_{\text{free}}(\lambda) . \quad (7.7)$$

(Strictly speaking this applies in the previous example too, although there the fermions were ultimately irrelevant.) The interaction term (5.3) also remains real, becoming

$$L_{\text{int}} = -m\bar{\lambda}\lambda \cos \rho \quad (7.8)$$

when expressed in terms ρ and λ . (See [21] for the super sine-Gordon S-matrix.)

When we consider other superalgebra models, we find that the situation encountered for bosonic theories may be reversed: starting from a model with an indefinite lagrangian, it may be possible to choose reality conditions which render the lagrangian positive-definite. The superalgebra models that have positive-definite actions with conventional reality conditions were studied by Olshanetsky [9]; these are the theories to which we have already drawn attention in Table 2. In addition, it was pointed out in [14] that the family of conformal models based on the algebras $C(m)$ with simple roots of type D_* have indefinite signature with conventional reality conditions, but become positive-definite when twisted reality conditions are used.

The first two members of this conformal series were discussed as examples (4) and (6) in Table 3 (recall that $A(1,0) = C(2)$). Referring to the explicit descriptions of these models and their Cartan matrices given there, we clearly have a reflection symmetry σ defined by exchange of the roots labelled in each case by 1 and 2. The corresponding reality condition (7.1) implies that u and χ should still be real, but now $v = iw$ and $\eta = i\lambda$ where w and λ are real. For the lagrangian written in (4.7) we see that L_+ is unchanged but $L_0 = -L_{\text{free}}(v) = L_{\text{free}}(w)$ so that L_{bos} becomes positive-definite. Similarly in the fermionic sector (4.9) we have $L_{\text{free}}(\eta) = -L_{\text{free}}(\lambda)$ so that L_{ferm} is now also positive-definite. Lastly, the interaction term (4.10) becomes

$$L_{\text{int}} = (\bar{\chi}\chi - \bar{\lambda}\lambda) \exp u \cos w - 2\bar{\chi}\lambda \exp u \sin w . \quad (7.9)$$

The case $C(2)$ is the $N = 2$ super-Liouville theory; for more details, including the $N = 2$ superspace description, see eg. [15,12].

In [14] it was shown that $C(m)$ is the only family of conformal theories for which twisted reality conditions lead to positive-definite actions in the way we have just described. The techniques we have developed in this paper allow one to prove this in a much simpler way, and also to extend the analysis beyond the super sinh/sine-Gordon theory to all affine superalgebras.

Proposition 8: The simple root systems \mathcal{R} which yield positive-definite Toda lagrangians in conjunction with twisted reality conditions are those listed in Table 5.

Table 5: Root systems with positive-definite twisted reality conditions

(a) Conformal theories

Superalgebra	Simple root system	Bosonic sector
$A(1, 0) = C(2)$	A^* 2 nodes	$A_1 \oplus U(1)$
$C(m+1)$	D_* $m+1$ nodes	$C_m \oplus U(1)$

(b) Massive theories

Superalgebra	Simple root system	Bosonic sector
$A_1^{(1)}$	B-B 2 nodes	$A_1^{(1)}$
$C(2)^{(2)}$	B^*-B^* 2 nodes	$A_1^{(1)}$
$A(1, 1)^{(1)}$	\widehat{A}^* 4 nodes	$A_1^{(1)} \oplus A_1^{(1)}$
$B(1, 1)^{(1)}$	D^*-B^* 3 nodes	$A_1^{(1)} \oplus A_1^{(1)}$
$D(2, 1)^{(1)}$	D^*-D^* 4 nodes	$A_1^{(1)} \oplus A_1^{(1)} \oplus A_1^{(1)}$
$D(2, 1; \alpha)^{(1)}$	—	$A_1^{(1)} \oplus A_1^{(1)} \oplus A_1^{(1)}$

(c) Mixed conformal and massive bosonic sectors

Superalgebra	Simple root system	Bosonic sector
$C(m)^{(1)}$	D^*-C $m+1$ nodes	$C_{m-1}^{(1)} \oplus U(1)$
$A(2m-1, 1)^{(2)}$	D^*-D $m+2$ nodes	$A_{2m-1}^{(2)} \oplus U(1)$
$A(2m-2, 1)^{(2)}$	D^*-B $m+1$ nodes	$A_{2m-2}^{(2)} \oplus U(1)$
$B(1, m)^{(1)}$	D^*-B_* $m+2$ nodes	$C_m^{(1)} \oplus U(1)$
$D(2, m)^{(1)}$	D^*-D_* $m+3$ nodes	$C_m^{(1)} \oplus U(1) \oplus U(1)$

For all entries in the table, $m \geq 2$. The notation used for Dynkin diagrams combines the building blocks introduced in Figures 2, 3 and 4.

Proof: The theory will have definite signature only if one of the systems \mathcal{B}_\pm is either empty or corresponds to $A_1^{(1)}$. On any other kind of bosonic subsystem, the symmetry must act trivially. Together these facts tightly constrain the possible Dynkin diagrams which need be considered. But even when these conditions are met, we must still check that the scalar fields are also given the correct signs by the choice of reality condition. These observations quickly narrow down the possibilities to those listed in Table 5, which can then be checked case by case.

Notice that the last two classical entries in section (b) of Table 5 are limiting cases of the last two infinite families in section (c). It is important to treat them separately, however, since they behave differently from the other members of their families. This is because there are more connected fermionic nodes in their Dynkin diagrams, and so their bosonic sectors, as found by our diagrammatic methods, deviate slightly from the general patterns found for the infinite series. In particular, the scalar fields corresponding to the $U(1)$ factors which occur for the infinite families are absent in these two special cases. This is in accordance with Proposition 6, which states that a theory is purely massive if it is based on an affine system of fermionic simple roots.

The families in section (c) of Table 5 may prove interesting to investigate in the future, since they involve a mixture of massive and massless degrees of freedom. Their structure,

including the interaction potentials, can easily be found explicitly by adapting what we have already said about the conformal cases based on $C(2)$, $C(3)$ and the generalization to $C(m)$. Of most immediate interest, however, are the models singled out in section (b) of Table 5. The first two cases are the sinh/sine-Gordon theory and its supersymmetric extension, which we discussed above. In addition, we find the examples (3) (4) and (5) from our list given earlier in Table 4. In each of these cases the new reality conditions correspond to a reflection symmetry in a horizontal axis through the diagrams drawn in Table 4, and the orientations of the diagrams for \mathcal{B}_\pm then reveal how this symmetry acts on these associated sets of bosonic roots.

In example (4) it is important that $\gamma = 1$ in order to fulfill the second criterion in (7.2) of Proposition 7. Using the notation introduced earlier, the new reality conditions correspond to setting $v = iw$ and $\eta = i\lambda$. The bosonic lagrangian becomes positive-definite because $L_- = -L_{\sinh}(v, m) = L_{\sin}(w, m)$ giving

$$L_{\text{bos}} = L_{\sinh}(u; m) + L_{\sin}(w; m) \quad (7.10)$$

so that we end up with independent sinh-Gordon and sine-Gordon theories. Similarly, in the fermionic sector $L_{\text{free}}(\eta) = -L_{\text{free}}(\lambda)$ so that

$$L_{\text{ferm}} = L_{\text{free}}(\eta) + L_{\text{free}}(\lambda) \quad (7.11)$$

is also positive-definite. The interaction term (5.9) becomes

$$L_{\text{int}} = m(\bar{\chi}\chi - \bar{\lambda}\lambda) \cosh u \cos w - 2m\bar{\chi}\lambda \sinh u \sin w \quad (7.12)$$

which is clearly real, as it should be. This is the $N = 2$ super sine/sinh-Gordon theory (see [23] for a discussion of the S-matrix).

Examples (3) and (5) are related, as we remarked earlier. Concentrating on the latter case, the new reality conditions are $v_j = iw_j$ and $\eta_j = i\lambda_j$ with $j = 1, 2$, where the new fields are all real. We then find a bosonic sector consisting of a single sinh-Gordon model and two sine-Gordon models:

$$L_{\text{bos}} = L_{\sinh}(u; 2m) + 2L_{\sin}(w_1; m) + 2L_{\sin}(w_2; m) , \quad (7.13)$$

while the fermionic lagrangian is also positive-definite:

$$L_{\text{ferm}} = L_{\text{free}}(\chi) + 2L_{\text{free}}(\lambda_1) + 2L_{\text{free}}(\lambda_2) , \quad (7.14)$$

and the interaction term (5.12) becomes

$$\begin{aligned} L_{\text{int}} = & m(\bar{\chi}\chi - \bar{\lambda}_1\lambda_1 - \bar{\lambda}_2\lambda_2) \{ \exp u \cos(w_1 + w_2) + \exp(-u) \cos(w_1 - w_2) \} \\ & - 2m\bar{\lambda}_1\lambda_2 \{ \exp u \cos(w_1 + w_2) - \exp(-u) \cos(w_1 - w_2) \} \\ & - 2m\bar{\chi}\lambda_1 \{ \exp u \sin(w_1 + w_2) - \exp(-u) \sin(w_1 - w_2) \} \\ & - 2m\bar{\chi}\lambda_2 \{ \exp u \sin(w_1 + w_2) + \exp(-u) \sin(w_1 - w_2) \} . \end{aligned} \quad (7.15)$$

The generalization to $D(2, 1; \alpha)^{(1)}$ will be considered in detail elsewhere [43]. It would be very interesting to find the S-matrices corresponding to these new models.

8. Concluding Remarks

In this paper we have attempted to develop a general and coherent approach to the analysis of Toda systems based on Lie superalgebras. The observation at the heart of the paper concerns the relationship between the simple root system \mathcal{R} for a superalgebra and the two associated bosonic root systems \mathcal{B}_{\pm} , which allows one to read off the bosonic content of the integrable model. As part of this discussion, we also hope to have clarified a number of issues regarding Lie superalgebras, simple root systems, and their Dynkin diagrams, which may be of interest quite independent of any connection to integrable systems.

As far as Toda theories themselves are concerned, one obvious problem is to extend the methods developed here to the non-abelian situation. There are also a number of new models we have found which would be very interesting to study further at the quantum level, particularly the supersymmetric combinations of sine/sinh-Gordon theories discussed in the last section.

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References

- [1] A.N. Leznov and M.V. Saveliev, Lett. Math. Phys. **3** (1979) 489; Commun. Math. Phys. **74** (1980) 11; Commun. Math. Phys. **89** (1983) 59; Acta. Appl. Math. **16** (1989) 1
- [2] P. Mansfield, Nucl. Phys. **B208** (1982) 277; **B222** (1983) 419
- [3] A.V. Mikhailov, M.A. Olshanetsky and A.M. Perelomov, Commun. Math. Phys. **79** (1981) 473
- [4] D.I. Olive and N. Turok, Nucl. Phys. **B257** (1986) 277; **B265** (1986) 469
- [5] P. Mansfield and B. Spence, Nucl. Phys. **B362** (1991) 294
- [6] P. Forgács, A. Wipf, J. Balog, L. Fehér and L. O’Raifeartaigh, Phys. Lett. **B227** (1989) 214; L. Feher, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Phys. Rep. **222** (1992) 1
- [7] P. Bouwknegt and K. Schoutens, Phys. Rep. **223** (1993) 183
- [8] A.E. Arinstein, V.A. Fateev and A.B. Zamalodchikov, Phys. Lett. **B87** (1979) 389; H.W. Braden, E. Corrigan, P.E. Dorey and Y. Sasaki, Nucl. Phys. **B338** (1990) 689; P. Christie and G. Mussardo, Int. J. Mod. Phys. **A5** (1990) 4581; G.W. Delius, M.T. Grisaru and D. Zanon, Nucl. Phys. **B382** (1992) 365
- [9] M.A. Olshanetsky, Commun. Math. Phys. **88** (1983) 63
- [10] D.A. Leites, M.V. Saveliev and V.V. Serganova, *Embeddings of $Osp(N/2)$ and associated non-linear supersymmetric equations*, in *Group Theoretical Methods in Physics* Vol. 1 (Proc. Third Yurmala Seminar USSR 22-24 May 1985), (VNU Science Press, Utrecht, 1986)
- [11] H.C. Liao and P. Mansfield, Nucl. Phys. **B344** (1990) 696; Phys. Lett. **B267** (1991) 188
- [12] H.C. Liao and P. Mansfield, Phys. Lett. **B252** (1991) 230; **B252** (1991) 237
- [13] J.M. Evans and T.J. Hollowood, Nucl. Phys. **B352** (1991) 723, erratum Nucl. Phys. **B382** (1992) 662
- [14] J.M. Evans, Nucl. Phys. **B390** (1993) 225
- [15] J.M. Evans and T.J. Hollowood, Phys. Lett. **B293** (1992) 100
- [16] H. Nohara and K. Mohri, Nucl. Phys. **B349** (1991) 253; S. Komata, K. Mohri and H. Nohara, Nucl. Phys. **B359** (1991) 168; H. Nohara, Ann. Phys. **241** (1992) 1
- [17] G.M.T. Watts, Nucl. Phys. **B361** (1991) 311; K. Ito, Int. J. Mod. Phys. **A7** (1992) 4885
- [18] T. Inami and K-I. Izawa, Phys. Lett. **B255** (1991) 521; F. Delduc, E. Ragoucy and P. Sorba, Commun. Math. Phys. **146** (1992) 403; L. Frappat, E. Ragoucy and P. Sorba, Commun. Math. Phys. **157** (1993) 499
- [19] J.M. Evans and J.O. Madsen, Phys. Lett. **B384** (1996) 131
- [20] C. Ahn, D. Bernard and A. LeClair, Nucl. Phys. **B346** (1990) 409
- [21] C. Ahn, Nucl. Phys. **B354** (1990) 57

- [22] T. Inami and H. Kanno, Commun. Math. Phys. **136** (1991) 519; Nucl. Phys. **B359** (1991) 201
- [23] K. Kobayashi and T. Uematsu, Phys. Lett. **B264** (1991) 107; **B275** (1992) 361
- [24] M.T. Grisaru, S. Penati and D. Zanon, Phys. Lett. **B253** (1991) 357;
G.W. Delius, M.T. Grisaru, S. Penati and D. Zanon, Nucl. Phys. **B359** (1991) 125;
Phys. Lett. **B256** (1991) 164;
M.T. Grisaru, S. Penati and D. Zanon, Nucl. Phys. **B369** (1992) 373
- [25] S. Penati and D. Zanon, Phys. Lett. **B288** (1992) 297;
A. Gualzetti, S. Penati and D. Zanon, Nucl. Phys. **B398** (1993) 622
- [26] S. Penati, M. Pernici and D. Zanon, Phys. Lett. **B309** (1993) 304;
S. Penati and D. Zanon, Nucl. Phys. **B436** (1995) 265
- [27] M.D. Freeman, Phys. Lett. **B261** (1991) 57
A. Fring, H.C. Liao and D.I Olive, Phys. Lett. **B266** (1991) 82
- [28] J.E. Humphries, *Introduction to Lie Algebras and Representation Theory*,
(Springer-Verlag, 1972);
J.P. Serre, *Complex Semi-Simple Lie Algebras*, (Springer-Verlag, 1986)
- [29] P. Goddard and D. Olive, Int. J. Mod. Phys. **A1** (1986) 303
- [30] V.G. Kac, *Infinite Dimensional Lie Algebras*, 3rd Edition, (CUP, 1990)
- [31] V.G. Kac, Adv. Math. **26** (1978) 8
- [32] V.G. Kac, Adv. Math. **30** (1978) 85
- [33] V.G. Kac, *Contravariant form for infinite-dimensional Lie algebras and superalgebras*,
in *Group Theoretical Methods in Physics XIII*, Springer Lecture Notes in Physics,
Vol. 94 (1979) 441
- [34] V.V. Serganova, Math. USSR Iz. **24** (1985) 539
- [35] J.W. van de Leur, Commun. Alg. **17(8)** (1989) 1815;
Contragredient Lie Superalgebras of Finite Growth, Thesis (1986) Utrecht University
- [36] L. Frappat, A. Sciarrino and P. Sorba, Commun. Math. Phys. **121** (1989) 457
- [37] L. Frappat, P. Sorba and A. Sciarrino, *Dictionary on Lie Superalgebras*,
hep-th/9607161; preprint ENSLAPP-AL-600/96, DSF-T-30/96
- [38] T. Eguchi and S-K. Yang, Phys. Lett. **B224** (1989) 373; **B235** (1990) 282;
T.J. Hollowood and P. Mansfield, Phys. Lett. **B226** (1989) 73;
A. LeClair, Phys. Lett. **B230** (1989) 103;
D. Bernard and A. LeClair, Phys. Lett. **B247** (1990) 309;
F.A. Smirnov, Nucl. Phys. **B337** (1990) 156; Int. J. Mod. Phys. **A4** (1989) 4213;
N. Yu. Reshetikhin and F.A. Smirnov, Commun. Math. Phys. **131** (1991) 157
- [39] G. Felder, Nucl. Phys. **B317** (1989) 215;
P. Bouwknegt, J. McCarthy and K. Pilch, Commun. Math. Phys. **131** (1990) 125
- [40] T.J. Hollowood, Nucl. Phys. **B384** (1992) 523; Int. J. Mod. Phys. **A8** (1993) 947
- [41] J.L. Cardy and G. Mussardo, Phys. Lett. **B225** (1989) 275
- [42] J.M. Evans and J.O. Madsen, Phys. Lett. **B389** (1996) 665
- [43] J.M. Evans and J.O. Madsen, *Quantum Integrability of Coupled $N=1$ Super Sine/Sinh-Gordon Theories and the Lie Superalgebra $D(2,1;\alpha)$* , Preprint DAMTP/97-147 US-FT/35-97; hep-th/9712227